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Steady compressible Navier–Stokes–Fourier system with temperature dependent viscosities

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joint papers with: A. Novotný, D. Jesslé (Toulon); O. Kreml (Praha)

1 System of equations in the steady regime

- Balance of mass

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \tag{1}$$

$\varrho(x)$: $\Omega \mapsto \mathbb{R}$ . . . density of the fluid
$\mathbf{u}(x)$: $\Omega \mapsto \mathbb{R}^3$ . . . velocity field

- Balance of momentum

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla p = \varrho \mathbf{f} \tag{2}$$

$\mathbb{S}$ . . . viscous part of the stress tensor (symmetric tensor)
$\mathbf{f}(x)$: $\Omega \mapsto \mathbb{R}^3$ . . . specific volume force (given)
p. . . pressure (scalar quantity)

- Balance of total energy

$$\operatorname{div}\left(\varrho E\mathbf{u}\right) + \operatorname{div}\left(\mathbf{q} + p\mathbf{u}\right) = \varrho\mathbf{f}\cdot\mathbf{u} + \operatorname{div}\left(\mathbb{S}\mathbf{u}\right) \tag{3}$$

$E = \frac{1}{2}|\mathbf{u}|^2 + e$. . . specific total energy
e . . . specific internal energy (scalar quantity)
$\mathbf{q}$ . . . heat flux (vector field)
(no energy sources assumed)

2 Thermodynamics

We will work with basic quantities: density ϱ and temperature ϑ

We assume: $e = e(\varrho, \vartheta)$, $p = p(\varrho, \vartheta)$

- Gibbs' relation

$$\frac{1}{\vartheta}\Big(De(\varrho,\vartheta) + p(\varrho,\vartheta)D\Big(\frac{1}{\varrho}\Big)\Big) = Ds(\varrho,\vartheta) \tag{4}$$

with $s(\varrho, \vartheta)$ the specific entropy.

The entropy fulfills

- Entropy balance

$$\operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \tag{5}$$

- Second law of thermodynamics

$$\sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \geq 0 \tag{6}$$

3 Constitutive relations

- Newtonian fluid

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\Big[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \tfrac{2}{3}\mathrm{div}\, \mathbf{u}\mathbf{I}\Big] + \xi(\vartheta)\mathrm{div}\, \mathbf{u}\mathbf{I} \tag{7}$$

$\mu(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+$,
$\xi(\cdot)$: $\mathbb{R}^+ \to \mathbb{R}^+_0$: viscosity coefficients

- Fourier's law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta \tag{8}$$

$\kappa(\cdot)$: $\mathbb{R}^+ \mapsto \mathbb{R}^+ \ldots$ heat conductivity

- Pressure law

$$\begin{aligned} p = p(\varrho, \vartheta) \quad &= \quad \varrho^{\gamma} + \varrho\vartheta \\ \text{or} \quad &= \quad (\gamma - 1)\varrho e(\varrho, \vartheta) \end{aligned} \tag{9}$$

(we will not consider the latter, due to additional technicalities)

- Internal energy

$$e(\varrho, \vartheta) = c_v \vartheta + \frac{\varrho^{\gamma-1}}{\gamma - 1} \tag{10}$$

- Heat conductivity

$$\kappa(\vartheta) \sim (1 + \vartheta^m) \tag{11}$$

$m \in \mathbb{R}^+$

- Viscosity coefficients

$$\begin{gathered} C_1(1+\vartheta)^\alpha \leq \mu(\vartheta) \leq C_2(1+\vartheta)^\alpha \\ 0 \leq \xi(\vartheta) \leq C_2(1+\vartheta)^\alpha \end{gathered} \tag{12}$$

$\mu(\cdot)$ global Lipschitz continuous, $\xi(\cdot)$ continuous,
$0 \leq \alpha \leq 1$

4 Classical formulation of the problem

We consider steady solutions in a bounded domain $\Omega \subset \mathbb{R}^3$:

Steady compressible Navier–Stokes–Fourier system

$$\operatorname{div}(\varrho \mathbf{u}) = 0$$

$$\begin{aligned} &\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{u}) + \nabla p(\varrho, \vartheta) = \varrho \mathbf{f} \\ &\operatorname{div}\Big(\varrho\big(\frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta)\big)\mathbf{u}\Big) - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) \\ &= \operatorname{div}\big(-p(\varrho, \vartheta)\operatorname{div}\mathbf{u} + \mathbb{S}(\vartheta, \nabla \mathbf{u})\mathbf{u}\big) + \varrho \mathbf{f} \cdot \mathbf{u} \end{aligned} \tag{13}$$

Boundary conditions at $\partial\Omega$: velocity

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0 \\ (\mathbb{I} - \mathbf{n} \otimes \mathbf{n})(\mathbb{S}(\vartheta, \nabla \mathbf{u})\mathbf{n} + \lambda \mathbf{u}) &= \mathbf{0}, \end{aligned} \tag{14}$$

$\lambda \geq 0$

Boundary conditions at $\partial\Omega$: temperature

$$\kappa(\vartheta)\frac{\partial \vartheta}{\partial \mathbf{n}} + L(\vartheta - \Theta_0) = 0, \tag{15}$$

$L > 0$

Total mass

$$\int_{\Omega} \varrho \, \mathrm{d}x = M > 0 \tag{16}$$

Instead of total energy balance we can consider the entropy balance

Entropy balance

$$\begin{aligned} \operatorname{div} (\varrho s(\varrho, \vartheta)\mathbf{u}) - \operatorname{div} \left(\kappa(\vartheta) \frac{\nabla \vartheta}{\vartheta} \right) = \sigma \\ = \frac{\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}}{\vartheta} + \frac{\kappa(\vartheta)|\nabla \vartheta|^2}{\vartheta^2} \end{aligned} \tag{17}$$

5 Weak solution to our problem

- Weak formulation of the continuity equation

$$\int_{\Omega} \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \qquad \forall \psi \in C^1(\overline{\Omega}) \tag{18}$$

- Renormalized continuity equation

ϱ extended by zero outside Ω, $\mathbf{u}$ extended outside Ω so that it remains in the $W^{1,p}$ space

$$\int_{\Omega} b(\varrho)\mathbf{u}\cdot\nabla\psi \, \mathrm{d}x + \int_{\Omega} \big(\varrho b'(\varrho) - b(\varrho)\big) \operatorname{div} \mathbf{u} \psi \, \mathrm{d}x = 0 \forall \psi \in C_0^1(\mathbb{R}^3) \tag{19}$$

for all $b \in C^1([0,\infty)) \cap W^{1,\infty}(0,\infty)$ with $zb'(z) \in L^\infty(0,\infty)$

- Weak formulation of the momentum equation

$$\int_\Omega \big(- \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} - p(\varrho, \vartheta) \mathrm{div}\, \boldsymbol{\varphi} + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \boldsymbol{\varphi} \big) \, \mathrm{d}x$$
$$+ \lambda \int_{\partial\Omega} \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}\sigma = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \quad \forall \boldsymbol{\varphi} \in C^1_{\mathbf{n}}(\overline{\Omega}; \mathbb{R}^3) \tag{20}$$

- Weak formulation of the total energy balance

$$
\begin{aligned}
&\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x \\
&= \int_\Omega \big(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x \\
&- \int_\Omega \big((\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u})\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\big)\,\mathrm{d}x \\
&- \int_{\partial\Omega} (L(\vartheta-\Theta_0) + \lambda|\mathbf{u}|^2)\psi\,\mathrm{d}\sigma \\
&\forall\psi\in C^1(\overline{\Omega})
\end{aligned}
\tag{21}
$$

Definition 1. *The triple* $(\varrho,\mathbf{u},\vartheta)$ *is called a renormalized weak solution to our system (13)–(16) if* $\varrho\geq 0$, $\vartheta > 0$, $\mathbf{u}\cdot\mathbf{n}=0$, $\int_\Omega \varrho \,\mathrm{d}x = M$, *(18), (19), (20) and (21) hold true.*

6 Entropy variational solution to our problem

- Weak formulation of the entropy inequality

$$\int_\Omega \Big(\frac{\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}}{\vartheta} + \kappa(\vartheta)\frac{|\nabla \vartheta|^2}{\vartheta^2}\Big)\psi \,\mathrm{d}x + \int_{\partial\Omega} \frac{L}{\vartheta}\Theta_0 \psi \,\mathrm{d}\sigma$$
$$\leq \int_{\partial\Omega} L\psi \,\mathrm{d}\sigma + \int_\Omega \Big(\kappa(\vartheta)\frac{\nabla\vartheta \cdot \nabla\psi}{\vartheta} - \varrho s(\varrho, \vartheta)\mathbf{u} \cdot \nabla\psi\Big)\,\mathrm{d}x$$
$$\forall \text{ nonnegative } \psi \in C^1(\overline{\Omega}) \tag{22}$$

- Global total energy balance

$$\int_{\partial\Omega} (L(\vartheta - \Theta_0) + \lambda|\mathbf{u}|^2)\,\mathrm{d}\sigma = \int_{\Omega} \varrho \mathbf{f} \cdot \mathbf{u}\,\mathrm{d}x \qquad (23)$$

Definition 2. *The triple* $(\varrho, \mathbf{u}, \vartheta)$ *is called a renormalized variational entropy solution to our system (13)–(16), if* $\varrho \geq 0$, $\vartheta > 0$, $\mathbf{u} \cdot \mathbf{n} = 0$, $\int_\Omega \varrho \ \mathrm{d}x = M$ *(18), (19) and (20) are satisfied in the same sense as in Definition 1, and we have the entropy inequality (22) together with the global total energy balance (23).*

Both definitions are reasonable in the sense that any smooth weak or entropy variational solution is actually a classical solution to (13)–(16).

7 Mathematical results

Until 2009, in the literature there was no existence results except for small data results or one result by P.L. Lions, where, however, the fixed mass was replaced by the finite L^p norm of the density for p sufficiently large.

Mucha, M.P.: Commun. Math. Phys. (2009)

Assumptions: constant viscosity, $L(\vartheta) \sim (1+\vartheta)^l$

Result: existence of weak solution (even fulfilling the internal energy equality) for $\gamma > 3$, $l+1 = m > \frac{3\gamma-1}{3\gamma-7}$, $\varrho \in L^\infty(\Omega)$, $\mathbf{u}, \vartheta \in W^{1,q}(\Omega)$, $q < \infty$

Mucha, M.P.: M3AS (2010)

Assumptions: constant viscosity, slip or homogeneous Dirichlet boundary conditions for the velocity, $L(\vartheta) \sim (1+\vartheta)^l$

Result: existence of weak solution for $\gamma > \frac{7}{3}$, $l + 1 = m > \frac{3\gamma-1}{3\gamma-7}$

Novotný, M.P.: J. Differential Equations (2011)

Assumptions: viscosity temperature dependent $\mu(\vartheta), \xi(\vartheta) \sim (1+\vartheta)$ $(\alpha = 1)$, $L \sim const$ $(l = 0)$ homogeneous Dirichlet condition for the velocity (but slip b.c. can be treated via the same method)

Result: existence of an entropy variational solution for $\gamma > \frac{3}{2}$ and $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\}$, existence of a weak solution if additionally $\gamma > \frac{5}{3}$, $m > 1$

Comments: the estimates for the density deduced from Bogovskii type estimates, which leads to $\gamma > \frac{3}{2}$; better result with respect to the constant viscosity due to the entropy inequality

Novotný, M.P.: SIAM J. Math. Anal. (2011)

Assumptions: the same as in the previous case.

Result: existence of an entropy variational solution for $\gamma > \frac{3+\sqrt{41}}{8}$ and $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}, \frac{2\gamma(4\gamma-1)}{9(4\gamma^2-3\gamma-2)}\}$, existence of a weak solution if additionally $\gamma > \frac{4}{3}$, $m > \max\{1, \frac{2\gamma}{3(3\gamma-4)}\}$

Comments: Improvement is achieved due additional local pressure estimates; following the idea of Frehse, Steinhauer, Weigant (used for the Navier–Stokes equations), we are able to get additional estimates for the density of the form

$$\sup_{y\in\overline{\Omega}} \int_\Omega \frac{p(\varrho,\vartheta)}{|x-y|^A}\,\mathrm{d}x < +\infty$$

with $A = A(m)$. This gives the a priori estimates for any $\gamma > 1$. The lower bound for γ comes from the limit passage.

8 Case $\alpha = 1$

We now present the new results from the paper Jesslé, Novotný, M.P. (2012). We consider the case $\alpha = 1$ and for simplicity Ω not axially symmetric and $\lambda = 0$. We have in this case Korn's inequalities of the type

$$\|\mathbf{u}\|_{1,2}^2 \leq C \begin{cases} \displaystyle\int_\Omega \frac{1}{\vartheta}\mathbb{S}(\vartheta,\mathbf{u}) : \nabla\mathbf{u}\ \mathrm{d}x \\ \displaystyle\int_\Omega \mathbb{S}(\vartheta,\mathbf{u}) : \nabla\mathbf{u}\ \mathrm{d}x. \end{cases}$$

We assume we are at the last step in the approximative scheme, i.e. for $\delta > 0$ and β, B sufficiently large we have the existence of the triple $(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$ fulfilling

Continuity equation:

$$\int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \nabla \psi \ \mathrm{d}x = 0 \tag{24}$$

for all $\psi \in W^{1,\frac{30\beta}{25\beta-18}}(\Omega;\mathbb{R})$, as well as in the renormalized sense

Momentum equation:

$$\begin{aligned}\int_\Omega \Big(-\varrho_\delta(\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla\boldsymbol{\varphi} + \mathbb{S}(\vartheta_\delta, \nabla\mathbf{u}_\delta) : \nabla\boldsymbol{\varphi} \\ -\big(p(\varrho_\delta,\vartheta_\delta) + \delta\varrho_\delta^\beta + \delta\varrho_\delta^2\big)\operatorname{div}\boldsymbol{\varphi}\Big) \ \mathrm{d}x = \int_\Omega \varrho_\delta \mathbf{f}\cdot\boldsymbol{\varphi} \ \mathrm{d}x\end{aligned} \tag{25}$$

for all $\boldsymbol{\varphi} \in W_{\mathbf{n}}^{1,\frac{5}{2}}(\Omega;\mathbb{R}^3)$

Total energy balance:

$$
\begin{aligned}
&\int_\Omega \Big(\big(-\frac{1}{2}\varrho_\delta|\mathbf{u}_\delta|^2 - \varrho_\delta e(\varrho_\delta,\vartheta_\delta)\big)\mathbf{u}_\delta\cdot\nabla\psi \\
&+\big(\kappa(\vartheta_\delta)+\delta\vartheta_\delta^B+\delta\vartheta_\delta^{-1}\big)\nabla\vartheta_\delta\cdot\nabla\psi\Big)\ \mathrm{d}x \\
&+\int_{\partial\Omega}\big(L+\delta\vartheta_\delta^{B-1}\big)(\vartheta_\delta-\Theta_0)\psi\,\mathrm{d}\sigma \\
&=\int_\Omega \varrho_\delta\mathbf{f}\cdot\mathbf{u}_\delta\psi\ \mathrm{d}x+\int_\Omega\Big(\big(-\mathbb{S}(\vartheta_\delta,\nabla\mathbf{u}_\delta)\mathbf{u}_\delta \\
&+\big(p(\varrho_\delta,\vartheta_\delta)+\delta\varrho_\delta^\beta+\delta\varrho_\delta^2\big)\mathbf{u}_\delta\big)\cdot\nabla\psi+\delta\vartheta_\delta^{-1}\psi\Big)\ \mathrm{d}x \\
&+\delta\int_\Omega\Big(\frac{1}{\beta-1}\varrho_\delta^\beta+\varrho_\delta^2\Big)\mathbf{u}_\delta\cdot\nabla\psi\ \mathrm{d}x
\end{aligned}
\tag{26}
$$

for all $\psi\in C^1(\overline{\Omega};\mathbb{R})$

Entropy inequality:

$$\int_\Omega \Big(\vartheta_\delta^{-1} \mathbb{S}(\vartheta_\delta, \mathbf{u}) : \nabla \mathbf{u}_\delta + \delta \vartheta_\delta^{-2} + \big(\kappa(\vartheta_\delta) + \delta \vartheta_\delta^B + \delta \vartheta_\delta^{-1} \big) \frac{|\nabla \vartheta_\delta|^2}{\vartheta_\delta^2} \Big) \psi \ \mathrm{d}x$$
$$\leq \int_\Omega \Big(\big(\kappa(\vartheta_\delta) + \delta \vartheta_\delta^B + \delta \vartheta_\delta^{-1} \big) \frac{\nabla \vartheta_\delta : \nabla \psi}{\vartheta_\delta} - \varrho s(\varrho_\delta, \vartheta_\delta) \mathbf{u}_\delta \cdot \nabla \psi \Big) \ \mathrm{d}x + \int_{\partial\Omega} \frac{L + \delta \vartheta_\delta^{B-1}}{\vartheta_\delta} (\vartheta_\delta - \Theta_0) \psi \, \mathrm{d}\sigma, \tag{27}$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$ nonnegative

8.1 Estimates independent of δ

Use in the entropy inequality and in the total energy balance test functions $\psi \equiv 1$:

$$
\begin{aligned}
&\int_\Omega \left(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1}\right)\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\ \mathrm{d}x \\
&+ \int_\Omega \left(\frac{1}{\vartheta_\delta}\mathbb{S}(\vartheta_\delta, \mathbf{u}_\delta) : \nabla\mathbf{u}_\delta + \delta\vartheta_\delta^{-2}\right)\ \mathrm{d}x \\
&\qquad + \int_{\partial\Omega} \frac{L + \delta\vartheta_\delta^{B-1}}{\vartheta_\delta}\Theta_0\,\mathrm{d}\sigma \\
&\qquad \leq \int_{\partial\Omega} \left(L + \delta\vartheta_\delta^{B-1}\right)\mathrm{d}\sigma.
\end{aligned}
\tag{28}
$$

$$\int_{\partial\Omega} \left(L\vartheta_\delta + \delta\vartheta_\delta^B\right) \mathrm{d}\sigma = \int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \mathbf{f} \ \mathrm{d}x$$
$$+ \int_{\partial\Omega} \left(L + \delta\vartheta_\delta^{B-1}\right) \Theta_0 \,\mathrm{d}\sigma + \delta \int_\Omega \vartheta_\delta^{-1} \ \mathrm{d}x \tag{29}$$

Using suitable estimates of the Bogovskii-type we can get rid of the δ-dependent terms and we conclude:

$$\|\mathbf{u}_\delta\|_{1,2} + \|\nabla \vartheta_\delta^{\frac{m}{2}}\|_2 + \|\nabla \ln \vartheta_\delta\|_2 + \|\vartheta_\delta^{-1}\|_{1,\partial\Omega}$$
$$+\delta\left(\|\nabla\vartheta_\delta^{\frac{B}{2}}\|_2^2 + \|\nabla\vartheta_\delta^{-\frac{1}{2}}\|_2^2 + \|\vartheta_\delta\|_{3B}^B + \|\vartheta_\delta^{-2}\|_1\right) \leq C \tag{30}$$

$$\|\vartheta_\delta\|_{3m} + \delta\|\vartheta_\delta\|_{B,\partial\Omega}^B \leq C\left(1 + \|\mathbf{u}_\delta \varrho_\delta\|_1\right) \tag{31}$$

8.2 Local estimates of the pressure

Denote for $1 \leq a \leq \gamma$, $0 < b < 1$

$$\mathcal{A} = \int_\Omega (\varrho_\delta^a |\mathbf{u}_\delta|^2 + \varrho_\delta^b |\mathbf{u}_\delta|^{2b+2}) \ \mathrm{d}x \tag{32}$$

We have

$$\|\mathbf{u}_\delta\|_{1,2} \leq C$$
$$\|\vartheta_\delta\|_{3m} \leq C\Big(1 + \mathcal{A}^{\frac{a-b}{2(ab+a-2b)}}\Big)$$
$$\int_\Omega \Big(\varrho_\delta^{s\gamma} + \varrho_\delta^{(s-1)\gamma} p(\varrho_\delta, \vartheta_\delta) + (\varrho_\delta |\mathbf{u}_\delta|^2)^s + \delta \varrho_\delta^{\beta+(s-1)\gamma}\Big) \ \mathrm{d}x$$
$$\leq C(1 + \mathcal{A}^{\frac{sa-b}{ab+a-b}}), \tag{33}$$

provided $1 < s < \frac{1}{2-a}$, $0 < (s-1)\frac{a}{a-1} < b < 1$, $s \leq \frac{6m}{3m+2}$, $m > \frac{2}{3}$. The last estimate follows from Bogovskii type estimates.

Next we want to use of the test functions of the type

$$\varphi_i(x) = \frac{(x-y)_i}{|x-y|^A}.$$

We have to work separately near the boundary and in the interior.

Lemma 1. *Let* $y \in \Omega$, $R_0 < \frac{1}{3} dist\,(y, \partial\Omega)$. *Then*

$$\int_{B_{R_0}(y)} \Big(\frac{p(\varrho_\delta, \vartheta_\delta)}{|x-y|^A} + \frac{\varrho_\delta |\mathbf{u}_\delta|^2}{|x-y|^A} \Big) \, \mathrm{d}x \\ \leq C \Big(1 + \|p(\varrho_\delta, \vartheta_\delta)\|_1 + \|\mathbf{u}_\delta\|_{1,2}(1 + \|\vartheta_\delta\|_{3m}) + \|\varrho_\delta |\mathbf{u}_\delta|^2\|_1 \Big), \tag{34}$$

provided $A < \min\left\{\frac{3m-2}{2m}, 1\right\}$.

Proof. We use as test function in the approximative momentum balance

$$\varphi_i(x) = \frac{(x-y)_i}{|x-y|^A}\tau^2$$

with $\tau \equiv 1$ in $B_{R_0}(y)$, R_0 as above, $\tau \equiv 0$ outside $B_{2R_0}(y)$, $|\nabla\tau| \le \frac{C}{R_0}$. Note that

$$\operatorname{div}\boldsymbol{\varphi} = \frac{3-A}{|x-y|^A}\tau^2 + g_1(x),$$

$$\partial_i\varphi_j = \Big(\frac{\delta_{ij}}{|x-y|^A} - A\frac{(x-y)_i(x-y)_j}{|x-y|^{A+2}}\Big)\tau^2 + g_2(x)$$

with g_1, g_2 in $L^\infty(\Omega)$. Thus we get the estimates from the pressure term and the convective term. We control the elliptic term provided

$\frac{1}{q} = 1 - \frac{1}{2} - \frac{1}{3m} > \frac{A}{3}$, implying $A < \frac{3m-2}{2m}$ for $m > \frac{2}{3}$. $\square$

Near the boundary, we use to use a similar test function. The test function due to Frehse, Steinhauer and Weigant, which can be used for both slip and no slip boundary conditions, leads to artificial restrictions on m. Assume for a moment that we deal with a flat part of the boundary which is described by $x_3 = 0$, i.e. $a(x') = 0$, $x' \in \mathcal{O} \subset \mathbb{R}^2$ with the normal vector $\mathbf{n} = (0, 0, -1)$ and $\mathbf{t}^1 = (1, 0, 0)$, $\mathbf{t}^2 = (0, 1, 0)$ the tangent vectors. Consider the points in the neighborhood of the origin. Then the test function which replaces the test function above can be taken in the form

$\mathbf{w}(x) = \mathbf{v}(x - y)$, where

$$\mathbf{v}(z) = \begin{cases} \dfrac{1}{|z|^A}(z_1, z_2, z_3) = (z \cdot \mathbf{t}^1)\mathbf{t}^1 + (z \cdot \mathbf{t}^2)\mathbf{t}^2 \\ +((0, 0, z_3 - a(z')) \cdot \mathbf{n})\mathbf{n}, z_3 \geq 0, \\ \dfrac{1}{|z|^A}(z_1, z_2, 0) = (z \cdot \mathbf{t}^1)\mathbf{t}^1 + (z \cdot \mathbf{t}^2)\mathbf{t}^2, z_3 < 0. \end{cases}$$

Note that if $y \in \overline{\Omega}$ (i.e. $y_3 \geq 0$), then $(\mathbf{w} \cdot \mathbf{n})(x) = w_3(x) = 0$ for $x_3 = 0$. For a general C^2 domain we use partition of unity and local flattening of the boundary. Therefore we get the same result as in Lemma 1 also in the neighborhood of the boundary, i.e. for any point in $\overline{\Omega}$.

We distinguish two cases. For $m \geq 2$ we have $\frac{3m-2}{2m} \geq 1$, hence $A < 1$ is the only restriction. If $m \in (\frac{2}{3}, 2)$, we have $A < \frac{3m-2}{2m}$.

For $m \geq 2$, passing $A \to 1^-$

Lemma 2. *Let $b \in ((s-1)\frac{\gamma}{\gamma-1}, 1)$, $1 < s < \frac{2}{2-\gamma}$, $m \geq 2$, $s \leq \frac{6m}{3m+2}$. Then there exists C independent of δ such that for any $y \in \overline{\Omega}$*

$$\int_\Omega \frac{p(\varrho_\delta, \vartheta_\delta) + (\varrho_\delta |\mathbf{u}_\delta|^2)^b}{|x-y|} \, \mathrm{d}x \leq C\big(1 + \delta\|\varrho_\delta\|_\beta^\beta + \|p(\varrho_\delta, \vartheta_\delta)\|_1 + (1 + \|\vartheta_\delta\|_{3m})\|\mathbf{u}_\delta\|_{1,2} + \|\varrho_\delta|\mathbf{u}_\delta|^2\|_1\big). \tag{35}$$

If $m < 2$, we take $1 \leq a < \gamma$ and relatively easily by Hölder's inequality

Lemma 3. *Let* $b \in ((s-1)\frac{\gamma}{\gamma-1}, 1)$*,* $1 < s < \frac{2}{2-\gamma}$*,* $A > \max\{\frac{3a-2\gamma}{a}, \frac{3b-2}{b}\}$*,* $m \in (\frac{2}{3}, 2)$*. Then there exists* C *independent of* δ *such that for any* $y \in \overline{\Omega}$

$$\int_{\Omega} \frac{\varrho_\delta^a + (\varrho_\delta |\mathbf{u}_\delta|^2)^b}{|x-y|} \, \mathrm{d}x$$
$$\leq C\Big(1 + \delta \|\varrho_\delta\|_\beta^\beta + \|p(\varrho_\delta, \vartheta_\delta)\|_1 + (1 + \|\vartheta_\delta\|_{3m}) \|\mathbf{u}_\delta\|_{1,2}$$
$$+ \|\varrho_\delta |\mathbf{u}_\delta|^2\|_1\Big)^{\max\{\frac{a}{\gamma}, b\}}. \tag{36}$$

Let us consider

$$-\Delta h = \varrho_\delta^a + \varrho_\delta^b |\mathbf{u}_\delta|^{2b} - \frac{1}{|\Omega|}\int_\Omega (\varrho_\delta^a + \varrho_\delta^b|\mathbf{u}_\delta|^{2b})\mathrm{d}x, \quad \frac{\partial h}{\partial \mathbf{n}}|_{\partial\Omega} = 0. \tag{37}$$

The unique strong solution can be written

$$h(y) = \int_\Omega G(x,y)(\varrho_\delta^a + \varrho_\delta^b|\mathbf{u}_\delta|^{2b})\,\mathrm{d}x + l.o.t.; \tag{38}$$

as $G(x,y) \le C|x-y|^{-1}$, we get

- $m \geq 2$

$$\|h\|_\infty \leq C(1 + \mathcal{A}^{\frac{\gamma - b/s}{b\gamma + \gamma - 2b}}), \tag{39}$$

provided

$$1 < s < \frac{1}{2-\gamma}, \quad 0 < (s-1)\frac{\gamma}{\gamma - 1} < b < 1, \quad s \leq \frac{6m}{3m+2} \tag{40}$$

- $\frac{2}{3} < m < 2$

$$\|h\|_\infty \leq C(1 + \mathcal{A}^{\frac{a-b/s}{ab+a-2b}\frac{a}{\gamma}} + \mathcal{A}^{\frac{a-b/s}{ab+a-2b}b}), \tag{41}$$

provided

$$1 < s < \frac{1}{2-a}, \quad 0 < (s-1)\frac{a}{a-1} < b < 1, \quad s \leq \frac{6m}{3m+2},$$
$$t > \frac{3a-2\gamma}{a}, \quad t > \frac{3b-2}{b}, \quad t < \frac{3m-2}{2m}. \tag{42}$$

Now

$$\mathcal{A} = \int_\Omega -\Delta h \mathbf{u}_\delta^2 \ \mathrm{d}x = \int_\Omega \nabla h \cdot \nabla |\mathbf{u}_\delta|^2 \ \mathrm{d}x \leq 2\|\nabla \mathbf{u}_\delta\|_2 B^{\frac{1}{2}}, \tag{43}$$

$$B = \int_\Omega |\nabla h \otimes \mathbf{u}_\delta|^2 \ \mathrm{d}x. \tag{44}$$

Employing once more integration by parts

$$B = -\int_\Omega h\Delta h|\mathbf{u}_\delta|^2 \, \mathrm{d}x - \int_\Omega h\nabla h \cdot \nabla \mathbf{u}_\delta \cdot \mathbf{u}_\delta \, \mathrm{d}x$$
$$\leq \|h\|_\infty(\mathcal{A} + \|\nabla \mathbf{u}_\delta\|_2 B^{\frac{1}{2}}),$$

i.e.

$$B \leq \|h\|_\infty \mathcal{A} + \frac{1}{2}\|\nabla \mathbf{u}_\delta\|_2^2 \|h\|_\infty^2. \tag{45}$$

Therefore

$$\mathcal{A} \leq C\|\nabla \mathbf{u}_\delta\|_2^2 \|h\|_\infty. \tag{46}$$

Hence,

$$\mathcal{A} \le C\big(1 + \mathcal{A}^{\frac{\gamma - b/s}{b\gamma + \gamma - 2b}}\big) \quad \text{if } m \ge 2,$$
$$\mathcal{A} \le C\big(1 + \mathcal{A}^{\frac{a-b/s}{ab+a-2b}\frac{a}{\gamma}} + \mathcal{A}^{\frac{a-b/s}{ab+a-2b}b}\big) \quad \text{if } \tfrac{2}{3} < m < 2, \tag{47}$$

Therefore

- $m \ge 2$

$$1 < s < \frac{1}{2-\gamma}, \quad 0 < (s-1)\frac{\gamma}{\gamma - 1} < b < 1,$$
$$s \le \frac{6m}{3m+2}, \quad \frac{\gamma - b/s}{b\gamma + \gamma - 2b} < 1 \tag{48}$$

- $m \in (\frac{2}{3}, 2)$

$$1 < s < \frac{1}{2-a}, \quad 0 < (s-1)\frac{a}{a-1} < b < 1, \quad s \le \frac{6m}{3m+2},$$
$$t > \frac{3a-2\gamma}{a}, \quad t > \frac{3b-2}{b}, \quad t < \frac{3m-2}{2m},$$
$$\frac{a-b/s}{ab+a-2b}\frac{a}{\gamma} < 1, \quad \frac{a-b/s}{ab+a-2b}b < 1. \tag{49}$$

Analyzing the conditions above we have

Lemma 4. *Let* $(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$ *solve our approximate problem. Let* $\gamma > 1$ *and* $m > \frac{2}{4\gamma-3}$.

Then there exists $s > 1$ *such that*

$$\begin{array}{lcr}
\sup_{\delta>0} \|\varrho_\delta\|_{\gamma s} & < & +\infty \\
\sup_{\delta>0} \|\varrho_\delta \mathbf{u}_\delta\|_{s} & < & +\infty \\
\sup_{\delta>0} \|\varrho_\delta |\mathbf{u}_\delta|^2\|_{s} & < & +\infty \\
\sup_{\delta>0} \|\mathbf{u}_\delta\|_{1,2} & < & +\infty \\
\sup_{\delta>0} \|\vartheta_\delta\|_{3m} & < & +\infty \\
\sup_{\delta>0} \|\vartheta_\delta^{m/2}\|_{1,2} & < & +\infty \\
\sup_{\delta>0} \delta \|\varrho_\delta^{\beta+(s-1)\gamma}\|_{1} & < & +\infty.
\end{array} \tag{50}$$

Moreover, we can take $s > \frac{6}{5}$ *provided* $\gamma > \frac{5}{4}$, $m > \max\{1, \frac{2\gamma+10}{17\gamma-15}\}$.

8.3 Limit passage $\delta \to 0^+$:

Continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \ \mathrm{d}x = 0 \tag{51}$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$

Momentum equation

$$\int_\Omega \Big(- \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \boldsymbol{\varphi} - \overline{p(\varrho, \vartheta)} \, \mathrm{div} \, \boldsymbol{\varphi} \Big) \ \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \tag{52}$$

for all $\boldsymbol{\varphi} \in C^1_{\mathbf{n}}(\overline{\Omega}; \mathbb{R}^3)$

Entropy inequality

$$\int_\Omega \left(\vartheta^{-1} \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \right) \psi \ \mathrm{d}x$$

$$\leq \int_\Omega \left(\kappa(\vartheta) \frac{\nabla \vartheta : \nabla \psi}{\vartheta} - \overline{\varrho s(\varrho, \vartheta)} \mathbf{u} \cdot \nabla \psi \right) \ \mathrm{d}x \qquad (53)$$

$$+ \int_{\partial \Omega} \frac{L}{\vartheta} (\vartheta - \Theta_0) \psi \, \mathrm{d}\sigma,$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$, nonnegative

Global total energy balance

$$\int_{\partial\Omega} L(\vartheta - \Theta_0)\,\mathrm{d}\sigma = \int_{\Omega} \varrho\mathbf{f} \cdot \mathbf{u}\ \mathrm{d}x \tag{54}$$

(total energy balance with test function $\psi \equiv 1$)

Total energy balance

$$\int_\Omega \Big(\big(-\frac{1}{2}\varrho|\mathbf{u}|^2 - \overline{\varrho e(\varrho,\vartheta)}\big)\mathbf{u}\cdot\nabla\psi$$
$$+\kappa(\vartheta)\nabla\vartheta : \nabla\psi\Big)\ \mathrm{d}x$$
$$+\int_{\partial\Omega} \big(L(\vartheta-\Theta_0)\psi\,\mathrm{d}\sigma = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\ \mathrm{d}x \tag{55}$$
$$+\int_\Omega \big(-\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u} + \overline{p(\varrho,\vartheta)}\mathbf{u}\big)\cdot\nabla\psi\ \mathrm{d}x$$

for all $\psi \in C^1(\overline{\Omega};\mathbb{R})$. We can pass only in certain situations, when we have better a priori estimates! We need $s > \frac{6}{5}$ and $m > 1$.

We need to show strong convergence of the density!

8.4 Strong convergence of the density

8.4.1 Effective viscous flux

Using as test function $\zeta(x)\nabla\Delta^{-1}(1_\Omega T_k(\varrho_\delta))$ with $T_k(z) = kT(\frac{z}{k})$, $k \in N$ for

$$T(z) = \begin{cases} z \text{ for } 0 \leq z \leq 1, \\ \text{concave on } (0,\infty), \\ 2 \text{ for } z \geq 3, \end{cases}$$

in the approximative balance of momentum, and $\zeta(x)\nabla\Delta^{-1}(1_\Omega \overline{T_k(\varrho)})$

in its limit version we can deduce

$$\overline{p(\varrho,\vartheta)T_k(\varrho)} - \Big(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\Big)\overline{T_k(\varrho)\,\mathrm{div}\,\mathbf{u}}$$
$$= \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} - \Big(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\Big)\overline{T_k(\varrho)}\,\mathrm{div}\,\mathbf{u} \qquad (56)$$

a.e. in Ω.

8.4.2 Oscillation defect measure

We do not have L^2-bound on the density and thus we do not know whether the renormalized continuity equation for the limit holds. To get it, we introduce:

Oscillation defect measure

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](Q) = \sup_{k>1} \left(\limsup_{\delta \to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \,\mathrm{d}x \right) \tag{57}$$

We have

$$\begin{aligned} \varrho_\delta \rightharpoonup \varrho \qquad & \text{in } L^1(\Omega;\mathbb{R}), \\ \mathbf{u}_\delta \rightharpoonup \mathbf{u} \qquad & \text{in } L^p(\Omega;\mathbb{R}^3), \\ \nabla\mathbf{u}_\delta \rightharpoonup \nabla\mathbf{u} \qquad & \text{in } L^p(\Omega;\mathbb{R}^{3\times 3}) \end{aligned}$$

and

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](\Omega) < \infty \tag{58}$$

for $q > p'$, then the limit density and velocity satisfy the renormalized continuity equation.

Assuming $m > \max\{\frac{2}{3(\gamma-1)}, \frac{2}{3}\}$, it can be verified that (58) holds true with some $2 < q < \gamma + 1$. This is the point giving additional restriction $m > \frac{2}{3(\gamma-1)}$.

8.4.3 Strong convergence of the density

We also get

$$\limsup_{\delta\to 0^+}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}\ \mathrm{d}x \\ \leq C\int_\Omega \left(\overline{p(\varrho,\vartheta)T_k(\varrho)}-\overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\right)\ \mathrm{d}x, \tag{59}$$

$$\limsup_{\delta\to 0^+}\int_\Omega \frac{1}{1+\vartheta}|T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}\ \mathrm{d}x \\ \leq C\int_\Omega \frac{1}{1+\vartheta}\left(\overline{p(\varrho,\vartheta)T_k(\varrho)}-\overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\right)\ \mathrm{d}x. \tag{60}$$

As $(\varrho_\delta, \mathbf{u}_\delta)$ and $(\varrho, \mathbf{u})$ verify the renormalized continuity equation, we have:

$$\int_\Omega T_k(\varrho)\,\mathrm{div}\,\mathbf{u}\ \,\mathrm{d}x = 0$$

and

$$\int_\Omega T_k(\varrho_\delta)\,\mathrm{div}\,\mathbf{u}_\delta\ \,\mathrm{d}x = 0, \qquad \text{i.e.} \int_\Omega \overline{T_k(\varrho)\,\mathrm{div}\,\mathbf{u}}\ \,\mathrm{d}x = 0$$

To this aim, use

$$\mathrm{div}\,(b(\varrho)\mathbf{u}) + \big(\varrho b'(\varrho) - b(\varrho)\big)\mathrm{div}\,\mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3)$$

with

$$b(\varrho) = \varrho \int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z,$$

i.e. $\varrho b'(\varrho) - b(\varrho) = T_k(\varrho)$.

Using the effective viscous flux identity we get that

$$\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\Big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big)\ \mathrm{d}x \\ = \int_\Omega \big(T_k(\varrho) - \overline{T_k(\varrho)}\big)\,\mathrm{div}\,\mathbf{u}\ \mathrm{d}x. \tag{61}$$

As $\lim_{k\to\infty}\|T_k(\varrho)-\varrho\|_1 = \lim_{k\to\infty}\|\overline{T_k(\varrho)}-\varrho\|_1 = 0$, the definition of the oscillation defect measure together with (58)

$$\lim_{k\to\infty}\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\Big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big)\ \mathrm{d}x = 0.$$

Hence

$$\lim_{k\to\infty} \limsup_{\delta\to 0^+} \int_\Omega \frac{1}{1+\vartheta} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \ \mathrm{d}x = 0,$$

which implies

$$\lim_{k\to\infty} \limsup_{\delta\to 0^+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{q} \ \mathrm{d}x = 0$$

with some $q > 2$, the same as for the oscillation defect measure.

Hence, as

$$\|\varrho_\delta - \varrho\|_1 \leq \|\varrho_\delta - T_k(\varrho_\delta)\|_1 + \|T_k(\varrho_\delta) - T_k(\varrho)\|_1 + \|T_k(\varrho) - \varrho\|_1,$$

we have

$$\varrho_\delta \to \varrho \qquad \text{in } L^1(\Omega;\mathbb{R})$$

which implies

$$\varrho_\delta \to \varrho \quad \text{in } L^p(\Omega;\mathbb{R}) \quad \forall 1 \leq p < s\gamma.$$

The proof of strong convergence is finished.

We proved:

Theorem 1. *Let* $\Omega \in C^2$ *be a bounded domain in* $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ *a.e. at* $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. *Let* $\gamma > 1$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$.

Let Ω *be not axially symmetric. Then there exists a variational entropy solution to our problem. Moreover,* $(\varrho, \mathbf{u})$ *is a renormalized solution to the continuity equation.*

Additionally, if $m > 1$ *and* $\gamma > \frac{5}{4}$, *then the solution is a weak solution, i.e. also the weak formulation of the total energy balance is fulfilled.*

9 Case: $0 \leq \alpha < 1$

We now discuss the new results from the paper Kreml, M.P. (2013). We consider for simplicity $\boldsymbol{\Omega}$ not axially symmetric and $\lambda = 0$. We have in this case Korn's inequalities of the form

$$\|\mathbf{u}\|_{1,p} \leq C\Big(\int_\Omega \frac{1}{\vartheta}\mathbb{S}(\vartheta, \nabla\mathbf{u}) : \nabla\mathbf{u}\ \mathrm{d}x\Big)^{\frac{1}{2}} \|\vartheta\|_{3m}^{\frac{1-\alpha}{2}},$$

where $p = \frac{6m}{3m+1-\alpha} < 2$. Therefore we get similarly as above uniform estimates

$$\|\vartheta_\delta\|_{3m} \leq C(1 + \|\varrho_\delta \mathbf{u}_\delta\|_1)$$
$$\|\mathbf{u}_\delta\|_{1,p} \leq C(1 + \|\vartheta_\delta\|_{3m}^{\frac{1-\alpha}{2}}).$$

Recall: for $1 \le a \le \gamma$, $0 < b < 1$

$$\mathcal{A} = \int_\Omega (\varrho_\delta^a |\mathbf{u}_\delta|^2 + \varrho_\delta^b |\mathbf{u}_\delta|^{2b+2}) \ \mathrm{d}x \tag{62}$$

We have

$$\begin{gathered} \|\mathbf{u}_\delta\|_{1,2} \le C\Big(1 + \mathcal{A}^{\frac{(1-\alpha)(a-b)}{2(p(a-b)+2b(a-1))}}\Big) \\ \|\vartheta_\delta\|_{3m} \le C\Big(1 + \mathcal{A}^{\frac{a-b}{p(a-b)+2b(a-1)}}\Big) \\ \int_\Omega \big(\varrho_\delta^{s\gamma} + \varrho_\delta^{(s-1)\gamma} p(\varrho_\delta, \vartheta_\delta) + (\varrho_\delta |\mathbf{u}_\delta|^2)^s + \delta \varrho_\delta^{\beta+(s-1)\gamma}\big) \ \mathrm{d}x \\ \le C(1 + \mathcal{A}^{\frac{2(sa-b)}{p(a-b)+2b(a-1)}}), \end{gathered} \tag{63}$$

provided $1 \le a \le \gamma$, $1 < s < \frac{p}{2+p-2a}$, $0 < \frac{a(2s-p)}{2a-p} < b < 1$, $s \le \frac{6m}{3m+1+\alpha}$, $m > \frac{1+\alpha}{3}$. The last estimate follows from Bogovskii type estimates.

Next we want to use of the test functions of the type

$$\varphi_i(x) = \frac{(x-y)_i}{|x-y|^A}.$$

We have to work separately near the boundary and in the interior. The procedure is similar to the previous case, with only changes corresponding to the fact that $\alpha < 1$ and $p = p(\alpha, m) < 2$. We get as before

Lemma 5. *We have*

$$\sup_{y\in\overline{\Omega}} \int_\Omega \Big(\frac{p(\varrho_\delta,\vartheta_\delta)}{|x-y|^A} + \frac{\varrho_\delta|\mathbf{u}_\delta|^2}{|x-y|^A}\Big)\,\mathrm{d}x$$
$$\leq C\big(1 + \|p(\varrho_\delta,\vartheta_\delta)\|_1 + \|\mathbf{u}_\delta\|_{1,2}(1+\|\vartheta_\delta\|_{3m}) + \|\varrho_\delta|\mathbf{u}_\delta|^2\|_1\big), \tag{64}$$

provided $A < \min\left\{\frac{3m-1-\alpha}{2m}, 1\right\}$.

The main difference appears at the step when we want to deduce from the estimate above some information about the density, momentum and kinetic energy. The approach based on the L^2, theory, which used the properties of the Green function for a certain elliptic problem, is not possible to apply. We return to the idea used in the paper by Novotný, Březina which was based on certain properties of Bessel kernels. However, their approach was also based on the L^2 theory which we have to generalize to the L^p setting.

We will consider Bessel kernels G_α in space $\mathbb{R}^N$ which are defined for any real index α via the Fourier transform

$$G_\alpha(x) := \mathcal{F}^{-1}((1+|\xi|^2)^{-\frac{\alpha}{2}}) \tag{65}$$

It can be shown that G_α is radially decreasing symmetric convolution kernel which is real and positive. It has exponential decay at infinity and following asymptotics at zero

$$G_\alpha(x) \le C(\alpha, N)|x|^{\alpha-N} \quad \text{as } |x| \to 0, \quad \text{for } \alpha \in (0, N). \tag{66}$$

Especially for $\alpha = 1$ we have

$$G_1(x) = C|x|^{-1}K_1(|x|),$$

where $K_\nu(r)$ are the MacDonald functions, with the following properties

$$K_\nu(r) \sim Cr^{-\nu} \text{ as } r \to 0, \quad K_\nu(r) \sim Cr^{1/2}e^{-r} \text{ as } r \to +\infty$$

$$(K_\nu(r))' = \frac{\nu}{r}K_\nu(r) - K_{\nu+1}(r).$$

For derivatives of Bessel kernels (denote $r = |x|$)

$$\frac{\mathrm{d}}{\mathrm{d}r}G_\alpha(r) = Cr^{\frac{\alpha-N}{2}}K_{\frac{N-\alpha+2}{2}}(r).$$

Using the results on the correspondance of behaviour of the function and its Fourier transform it can be shown that

Lemma 6. *For* $1 < r < \frac{3}{2}$ *we have*

$$(G_1 * G_1^r)(x) \leq C|x|^{1-2r} \quad \textit{as} \quad |x| \to 0.$$

The Bessel potential space $L^{\alpha,p}$

$$L^{\alpha,p}(\mathbb{R}^N) := \left\{ \varphi = G_\alpha * f, f \in L^p(\mathbb{R}^N) \right\}$$

endowed with the norm

$$\|G_\alpha * f\|_{L^{\alpha,p}(\mathbb{R}^N)} := \|f\|_{L^p(\mathbb{R}^N)}.$$

It is well known that for $\alpha \in \mathbb{N}$

$$W^{\alpha,p}(\mathbb{R}^N) = L^{\alpha,p}(\mathbb{R}^N)$$

with equivalence of norms.

Our method relies on the following result:

Theorem 2. *Let G be radially decreasing convolution kernel and let $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. Then for $1 < p \leq q < \infty$ the following statements are equivalent:*

1) There is a constant A_1 *such that*

$$\left(\int_{\mathbb{R}^N} |G * f|^q \mathrm{d}\mu\right)^{\frac{1}{q}} \leq A_1 \|f||_{L^p}$$

for all $f \in L^p(\mathbb{R}^N)$.

2) There is a constant A_2 *such that*

$$\|G * \mu_K\|_{L^{p'}} \leq A_2 \mu(K)^{\frac{1}{q'}}$$

for all compact sets K.

Moreover the constants A_1, A_2 *are comparable, in fact we can choose* $A_1 = A_2$.

For components of our velocity field $u_\delta^i \in W^{1,p}(\Omega)$ we find unique $f^i \in L^p(\Omega)$ such that $E(u_\delta^i) = G_1 * f^i$, where $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ is a continuous extension operator. We apply the theorem above in the following way, the spirit is the same as in Březina, Novotný: We take $N = 3$, $p = q \in (1, 2)$, $\mathrm{d}\mu(x) = (\varrho_\delta^a + (\varrho_\delta |\mathbf{u}_\delta|^2)^b)(x)\,\mathrm{d}x$, $G = G_1$ and $f = f^i$ defined above. We define $\varrho_\delta = 0$ outside Ω.

Denoting $h = (\varrho_\delta^a + (\varrho_\delta |\mathbf{u}_\delta|^2)^b)$ we can show that

$$\int_{\mathbb{R}^3} |G_1 * h|_K|^{p'} \mathrm{d}x$$
$$\leq C(\Omega) \|(G_1 * G_1^{p'-1}) * h^{p'-1}|_K\|_{L^\infty(\mathbb{R}^3)} \|h\|_{L^1(K)}$$

and we denote $A_2^{p'} := C(\Omega)\|(G_1 * G_1^{p'-1}) * h^{p'-1}\|_{L^\infty(\Omega)}$. Thus the statement 1) of Theorem above is satisfied and we conclude

$$
\begin{aligned}
&\|(\varrho_\delta^a + (\varrho_\delta|\mathbf{u}_\delta|^2)^b)|\mathbf{u}_\delta|^p\|_{L^1(\Omega)} \\
&\sim \sum_{i=1}^{3} \int_{\mathbb{R}^3} |E(u_\delta^i)|^p(x)(\varrho_\delta^a + (\varrho_\delta|\mathbf{u}_\delta|^2)^b)(x)\mathrm{d}x \\
&\leq \sum_{i=1}^{3} A_2^p \|E(u_\delta^i)\|^p_{L^{1,p}(\mathbb{R}^3)} \leq C A_2^p \|\mathbf{u}_\delta\|^p_{W^{1,p}(\Omega)}.
\end{aligned}
$$

Using the lemma on the asymptotic behaviour of the convolution of Bessel kernels we have

$$A_2^{p'} \leq C \sup_{x_0 \in \overline{\Omega}} \int_\Omega \frac{(\varrho_\delta^a + (\varrho_\delta |\mathbf{u}_\delta|^2)^b)^{p'-1}}{|x - x_0|^{2p'-3}} \mathrm{d}x$$

provided $p' > \frac{5}{2}$.

Therefore, assuming

$$\max \left\{ \frac{\gamma(3-p) - aA}{\gamma(p-1) - a}, \frac{(3-p) - bA}{p-1-b} \right\} < 3$$

we get, combining estimates above

$$
\begin{aligned}
\mathcal{A} = \|(\varrho_\delta^a + (\varrho_\delta|\mathbf{u}_\delta|^2)^b)|\mathbf{u}_\delta|^p\|_{L^1(\Omega)} \le C A_2^p \|\mathbf{u}_\delta\|^p_{W^{1,p}(\Omega)} \\
\le C \left(\sup_{x_0\in\overline{\Omega}} \int_\Omega \frac{(\varrho_\delta^a + (\varrho_\delta|\mathbf{u}_\delta|^2)^b)^{\frac{1}{p-1}}(x)}{|x-x_0|^{\frac{3-p}{p-1}}} \mathrm{d}x \right)^{p-1} \|\mathbf{u}_\delta\|^p_{W^{1,p}(\Omega)} \\
\le C \|\mathbf{u}_\delta\|^p_{W^{1,p}(\Omega)} \left(\left(\sup_{x_0\in\overline{\Omega}} \int_\Omega \frac{\varrho_\delta^\gamma}{|x-x_0|^A} \right)^{\frac{a}{\gamma}} \right. \\
\left. + \left(\sup_{x_0\in\overline{\Omega}} \int_\Omega \frac{\varrho_\delta|\mathbf{u}_\delta|^2}{|x-x_0|^A} \right)^b \right)
\end{aligned}
$$

Therefore finally

$$\mathcal{A} \leq C\|\mathbf{u}_\delta\|^p_{W^{1,p}(\Omega)} \times$$

$$\Big(\Big(1 + \delta\|\varrho_\delta\|^\beta_\beta + \|p(\varrho_\delta, \vartheta_\delta)\|_1 + (1 + \|\vartheta_\delta\|^\alpha_{3m})\|\mathbf{u}_\delta\|_{1,p} + \|\varrho_\delta|\mathbf{u}_\delta|^2\|_1\Big)$$

$$+ \Big(1 + \delta\|\varrho_\delta\|^\beta_\beta + \|p(\varrho_\delta, \vartheta_\delta)\|_1 + (1 + \|\vartheta_\delta\|^\alpha_{3m})\|\mathbf{u}_\delta\|_{1,p} + \|\varrho_\delta|\mathbf{u}_\delta|^2\|_1\Big)$$

and we may proceed as in the case with $\alpha = 1$, with only more complicated set of conditions on the parameters $\gamma < 1$, $m > 0$ and $\alpha \in [0, 1]$.

Let us only mention under which conditions we get a solution for special values of α. First, for $\alpha = 1$ we reobtain the result from the previous section. Next, for

$\alpha = \frac{1}{2}$:

Weak solution:

$$m \in \left(\tfrac{18+\sqrt{409}}{34}, \tfrac{3}{2}\right] \quad \gamma > \frac{(6m+3)(10m-1)}{6m^2-72m-5}$$

$$m \in \left(\tfrac{3}{2}, \infty\right) \quad \gamma > \frac{2m(10m-1)}{16m^2-18m+1}$$

Variational entropy solution:

$$m \in \left(\tfrac{5}{6}, \tfrac{3}{2}\right] \quad \gamma > \frac{6m(6m+3)}{48m^2-30m-3}$$

$$m \in \left(\tfrac{3}{2}, \tfrac{8}{3}\right] \quad \gamma > \frac{3m}{3m-2}$$

$$m \in \left(\tfrac{8}{3}, \infty\right) \quad \gamma > 1 + \frac{8}{9m}$$

$\alpha = 0$:

Weak solution:

$$m \in \left(\frac{9+\sqrt{65}}{8}, \frac{9+\sqrt{77}}{2}\right] \quad \gamma > \frac{m(5m-1)}{4m^2-9m+1}$$
$$m \in \left(\frac{9+\sqrt{77}}{2}, \infty\right) \quad \gamma > \frac{4m^2+8m-1}{3m^2}$$

Variational entropy solution:

$$m \in \left(\frac{5}{3}, \frac{8}{3}\right] \quad \gamma > \frac{3m}{3m-4}$$
$$m \in \left(\frac{8}{3}, \infty\right) \quad \gamma > 1 + \frac{8}{3m}$$

Generally, for m sufficiently large we have the following

Existence of weak solution:

$$\alpha \in [0, \tfrac{1}{3}] \quad \gamma > \tfrac{3\alpha+4}{3\alpha+3}$$
$$\alpha \in (\tfrac{1}{3}, 1] \quad \gamma > \tfrac{5}{4}$$

Existence of variational entropy solution:

$$\alpha \in [0, 1] \quad \gamma > 1$$

THANK YOU FOR YOUR ATTENTION!