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Properties and Applications of (max,min)-linear Equations and Inequalities..

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CONTENT:

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- Problem Formulation - General Scheme
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Motivation I.

- $\boxed{\text{i}} \longrightarrow \boxed{\text{j}} \longrightarrow \boxed{\text{T}}$,
- $\boxed{\text{T}} \longrightarrow \boxed{\text{j}} \longrightarrow \boxed{\text{i}}$.
- $i \in I,\ j \in J,\ I,\ J$ finite index sets,
- "quality level" a_{ij} of $\boxed{\text{i}} \longrightarrow \boxed{\text{j}}$,
- "quality level" b_{ij} of $\boxed{\text{j}} \longrightarrow \boxed{\text{i}}$,
- "quality level" $x_j =?$ of $\boxed{\text{j}} \longrightarrow \boxed{\text{T}}$
- "quality level" $y_j =?$ of $\boxed{\text{T}} \longrightarrow \boxed{\text{j}}$
- "quality levels" fuzzy values from $[0, 1]$.

Motivation I. continued

- Total "quality level" of
  $\boxed{\text{i}} \longrightarrow \boxed{\text{j}} \longrightarrow \boxed{\text{T}}$,
- is equal to $a_{ij} \wedge x_j \equiv \min(a_{ij}, x_j)$;
- Total "quality level" of
  $\boxed{\text{T}} \longrightarrow \boxed{\text{j}} \longrightarrow \boxed{\text{i}}$
- is equal to $b_{ij} \wedge y_j \equiv \min(b_{ij}, x_j)$;
- If $\boxed{\text{j}} \longrightarrow \boxed{\text{T}}$ is a two-way street, we have $x_j = y_j$.
- Levels $x_j,\ y_j$ are bounded variables in $[0, 1]$, i.e.
  $x_j \in [\underline{x}_j, \overline{x}_j] \subset [0, 1]$, $y_j \in [\underline{y}_j, \overline{y}_j] \subset [0, 1]$.
- $f_j(x_j),\ g_j(y_j)$ strictly monotone (increasing) or unimodal penalty functions (expenses).

Motivation I. - Continued.

- **Optimization Problem:**
- **Minimize** $f(x, y) \equiv \max(\max_{j \in J}(f_j(x_j)), \max_{j \in J}(g_j(y_j)))$
- **subject to**
- $\max_{j \in J}(a_{ij} \wedge x_j) \ R_i \ \max_{j \in J}(b_{ij} \wedge y_j) \ \ \forall i \in I,$
- $x_j \in [\underline{x}_j, \overline{x}_j], \ y_j \in [\underline{y}_j, \overline{y}_j].$

- where R_i is equal to one of the relations $\leq, \quad =, \quad \geq$.
- "One sided" constraints of the form:

$$\max_{j \in J}(a_{ij} \wedge x_j) \ R_i \ b_i \ \ \forall i \in I,$$

where $b_i \in (0, 1]$ are given can be obtained as a special case.

Motivation II. - Fuzzy Goals

- Let $I = \{1, \ldots, n\}, \; J = \{1, \ldots, n\}$;
- Let two groups of m fuzzy sets $A_i, \; B_i$ (fuzzy goals) be given; their membership functions are $\mu_i(j) = a_{ij}, \; \nu_i(j) = b_{ij}, \; j \in J$, $i \in I$;
- We have to find fuzzy set X with membership function $\mu_X(j), \; j \in J$, such that certain requirements concerning A_i, B_i, and X are fulfilled.
- Besides, we can look for optimal (in some sense) values $\mu_x(j), \; j \in J$ satifying the requirements.

Motivation II. - Fuzzy Goals

- Let for each $i \in I$, $\mu_{iX}(j) = a_{ij} \wedge x_j, \ \forall, \ j \in J$, $\nu_{iX}(j) = b_{ij} \wedge x_j, \ \forall \ j \in J$;

- For each $i \in I$, functions μ_{iX}, ν_{iX} are the membership functions of the fuzzy intersection of fuzzy sets $(A_i, \ X)$, $(B_i, \ X)$ respectively;

- We define for each $i \in I$ the heights of functions μ_{iX}, ν_{iX} as follows:

$$H_{A_iX}(\mu(j)) \equiv \max_{j \in J}(\mu_{iX}(j)),$$

$$H_{B_iX}(\nu(j)) \equiv \max_{j \in J}(\nu_{iX}(j)).$$

- We will assume that $f_j(\mu_X(j)), j \in J$ are given continuous increasing (in $\mu_X(j) = x_j$) penalty functions connected with the choice of $\mu_X(j)$;

Motivation II. - Fuzzy Goals - Continued

- **Example 1**
- $$\max_{j \in J} f_j(\mu_X(j)) \longmapsto \min$$
- subject to

$$H_{A_i X}(.) \geq b_i, \ \forall i \in I,$$

$$H_{B_i X}(.) \geq c_i, \ \forall i \in I,$$

where $b_i, \ c_i$ are given nonnegative numbers.

Motivation II. - Fuzzy Goals - Continued

- **Example 1 - reformulation**
-
$$\max_{j\in J} f_j(x_j) \longmapsto \min$$
- subject to
$$\max_{j\in J}(a_{ij} \wedge x_j) \geq b_i, \forall i \in I$$
$$\max_{j\in J}(b_{ij} \wedge x_j) \geq c_i, \forall i \in I$$
$$x_j \in [0,1] \; \forall j \in J$$

Motivation II. - Fuzzy Goals - Continued

- **Example 2**
-
$$\max_{j\in J} f_j(\mu_X(j)) \longmapsto \min$$
- subject to
$$H_{A_i}x(.) \;=\; H_{B_i}x(.),\; \forall i \in I$$

Motivation II. - Fuzzy Set Covering - Continued

- **Example 2 - reformulation**

- $$\max_{j \in J} f_j(x_j)) \longmapsto \min$$

- subject to

$$\max_{j \in J}(a_{ij} \wedge x_j) \ = \ \max_{j \in J}(a_{ij} \wedge x_j) \ \forall i \in I,$$

$$x_j \in [0,1] \ \forall j \in J.$$

General "Standard" Problem Formulation - Feasible Set.

- $J = \{1, \dots, n\}$, $I = \{1, \dots, m\}$,
  $R = (-\infty, +\infty)$, $R_+ = [0, +\infty)$,
- $R^n = R \times \dots \times R$ (n-times), $x^T = (x_1, \dots, x_n) \in R^n$,
  superscript T means transposition.
- $a_{ij},\ b_{ij}$ nonnegative $\forall i \in I$, $j \in J$ are given,
- $$a_i(x) \ \equiv \ \max_{j \in J}(a_{ij} \ o \ x_j) \quad \text{for all} \ \ i \in I\,,$$
- where o denotes one of the operations $\wedge,\ +,\ .$;
- $$b_i(x) \ \equiv \ \max_{j \in J}(b_{ij} \ o \ x_j) \quad \text{for all} \ \ i \in I\,,$$
- $M(\underline{x}, \overline{x}) \equiv \{x \in R^n \ ; \ a_i(x) = b_i(x) \ \ \forall i \in I,\ \underline{x} \leq x \leq \overline{x}\}$

General "Standard" Formulation - Optimization Problem.

- **
  - **Minimize** $f(x) \equiv \max_{j \in J} f_j(x_j)$
- **subject to**
  - $x \in M(\underline{x}, \overline{x})$.
- **
- where $f_j : [0,1] \to R,\ j \in J$ are continuous strictly increasing or unimodal functions.

Formulation of the Optimization Problem - Continued.

- In what follows we will consider the case, where $o = \wedge$, where $\alpha \wedge \beta \equiv \min(\alpha, \beta)$.
- i. e. we will consider the following problem:
- **Minimize f(x) subject to**
- $$\max(a_{ij} \wedge x_j) = \max_{j \in J}(b_{ij} \wedge x_j) \quad \forall i \in I,$$
- $$\underline{x} \leq x \leq \overline{x}$$

Properties of the Feasible Set.

- **Lemma 1**
- **(1) Let** $M(\overline{x}) \equiv \{x \in M\ ;\ a_i(x) = b_i(x),\ \ \forall i \in I,\ x \leq \overline{x}\}$**.** **Then** $M(\overline{x}) \neq \emptyset$**.**
- **(2) If** $M(\underline{x}, \overline{x}) \neq \emptyset$**, then there exists always its maximum element** $x^{\max}$**, i.e. there exists an element** $x^{\max} \in M(\underline{x}, \overline{x})$ **such that** $x \leq x^{\max}\ \ \forall x \in M(\underline{x}, \overline{x})$.
- **(3)** $M(\underline{x}, \overline{x}) \neq \emptyset$ **if and only if** $\underline{x} \leq x^{\max}$**.**

- **Remark**
- There exists a polynomial ($O(n^3)$) algorithm for finding $x^{\max}$ (see [Gavalec, Zimmermann, Kybernetika 2010]).

General Iteration Scheme - Application to (max, min)-linear Problems.

We apply the general iteration scheme to the problem

- **Minimize f(x)**
- **subject to**
- $\mathbf{x} \in \mathbf{M}(\underline{\mathbf{x}}, \overline{\mathbf{x}})$.
-

ALGORITHM I.

0. $\underline{f} := f(\underline{x}),\ \overline{f} := f(\overline{x})$;
1. Find the maximum element $x^{\max}$ of set $M(\overline{x})$;
2. If $\underline{x} \not\leq x^{\max}$, then $M(\underline{x}, \overline{x}) = \emptyset$, STOP;
3. $\alpha := f(x^{\max})$, $\underline{f}(\alpha) := \underline{f},\ \overline{f}(\alpha) := f(x^{\max})$;
4. $\alpha := \underline{f}(\alpha) + (\overline{f}(\alpha) - \underline{f}(\alpha))/2$, set $\overline{x}_j(\alpha) := f_j^{-1}(\alpha)\ \forall j \in J$;
5. Find $x^{\max}(\alpha) \in M(\overline{x}(\alpha))$;
6. If $\underline{x} \not\leq x^{\max}(\alpha)$, set $\underline{f}(\alpha) := \alpha$ go to 4;
7. If $f(x^{\max}(\alpha)) - \underline{f}(\alpha) < \epsilon$, set $x(\epsilon)^{opt} := x^{\max}(\alpha)$, STOP.
8. $\overline{f}(\alpha) := f(x^{\max}(\alpha))$, go to 4;

Extensions, Further Research.

- For $(\max, +)$-linear or $(\max, .)$-linear problems only pseudopolynomial algorithms [Bezem et al.].
- $(\min, \max)$- , $(\min, +)$- or $(\min, .)$-linear problems by analogy.
- For minimizing function $f(x)$ under one-sided constraints $\max_{j \in J}(a_{ij} \wedge x_j)\ R_i\ b_i\ \ \forall i \in I$,, there exists an exact algorithm see [Zimmermann, K. Theoretical computer Science 293(2003), pp.45 - 54].
- Exact (polynomial) optimization algorithms using the structure of the two-sided constraints is the subject of further research.
- Problems with objective function $f(x)$ and linear or convex constraints of the form: $M(\underline{x}, \overline{x}) \cap K$, where $K = \{x\ ;\ g_i(x_1, \ldots, x_n) \geq 0\}$, where $g_i,\ i \in I$ are concave functions and g_i are strictly increasing in all variables $x_j,\ j = 1, \ldots, n$.

References

- Bezem, M., Nieuwenhuis, R., Rodríguez-Carbonell, E.: Exponential Behaviour of the Butkovič - Zimmermann Algorithm for Solving Two-sided Linear Systems in Max-algebra, Discrete Applied Mathematics, 156(2008), pp. 3506-3509.
- Blagajewicz, M.: A New Approach for Global Optimization of MINLP Problems, contribution ALIO-INFORMS Joint International Meeting, June 2010, Buenos Aires.
- Gavalec, M., Zimmermann, K.: Solving Systems of Two-sided (max,min)-linear Equations, Kybernetika, 46 (2010), 405-414.
- Zimmermann, K.: Disjunctive Optimization Problems, Max-separable Problems and Extremal Algebras,Theoretical Computer Science, 293(2003), pp. 45 - 54 .

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