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**Streamlining the Applied Mathematics Studies
at Faculty of Science of Palacký University in Olomouc
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Department of Mathematical analysis
and Applications of Mathematics

Faculty of Science

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Free material optimization

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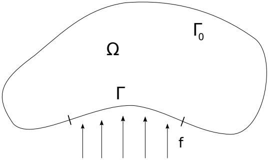
(a) homogenization method (M. Bendsoe, M. Kikuchi, A. V. Cherkaev, F. Allaire, J. H.)

(b) free material approach (M. Bendsoe, P. Pedersen, J. E. Taylor)

  (i) computational aspects (Ben-Tal, M. Kočvara, A. Nemirovski, J. Zowe)

  (ii) theoretical aspects

Setting of the problem



$$f \in L^2(\Gamma, \mathbb{R}^N), \ N \in \{2,3\}$$

$$(\widetilde{\mathcal{P}}(E)) \quad \begin{cases} \text{Find } u \text{ such that} \\ \begin{aligned} \operatorname{div} \sigma &= 0 && \text{in } \Omega \\ \sigma \cdot n &= f && \text{on } \Gamma \\ u_0 &= 0 && \text{on } \Gamma_0 \\ \sigma &= E \cdot \varepsilon(u) && \text{in } \Omega \end{aligned} \end{cases}$$

$$(\widetilde{\mathcal{P}}(E)) \quad \begin{cases} \text{Find } u \in V \text{ such that} \\ a_E(u, w) = \int_\Gamma f \cdot w \, ds \quad \forall w \in V, \end{cases}$$

where

$$V = \{v \in H^1(\Omega, \mathbb{R}^N) \mid v = 0 \text{ on } \Gamma_0\},$$

$$a_E(u, w) = \int_\Omega \langle E(x)\varepsilon(u(x)), \varepsilon(w(x))\rangle \, dx,$$

$$\langle E(x)\varepsilon(u(x)), \varepsilon(w(x))\rangle := E_{ijkl}\varepsilon_{ij}(u(x))\varepsilon_{kl}(w(x)).$$

Vectorial and the matricial representation of ε and E, respectively:

$$\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \sqrt{2}\varepsilon_{12})^T \in \mathbb{R}^3 \quad (\mathbb{R}^{\bar{N}}),$$

$$E = \begin{pmatrix} E_{1111} & E_{1122} & \sqrt{2}E_{1112} \\ & E_{2222} & \sqrt{2}E_{2212} \\ \text{sym.} & & 2E_{1212} \end{pmatrix} \in \mathbb{R}^{3\times 3} \ (\mathbb{R}^{\bar{N}\times\bar{N}}), \ \bar{N} = N(N+1)/2$$

The set of feasible materials

$$\mathcal{E}_0 = \{E \in L^\infty(\Omega, \mathbb{S}^{\bar{N}}) \mid E \succcurlyeq 0 \text{ a.e. in } \Omega\}$$

The set of admissible materials

$$\mathcal{E} = \{E \in \mathcal{E}_0 \mid \mathrm{Tr}\,(E) \le \bar{\rho} \text{ a.e. in } \Omega,\ v(E) \le \bar{v}\},$$

where

$$v(E) = \int_\Omega \mathrm{Tr}\,(E)\,dx \quad \text{and} \quad \bar{\rho}, \bar{v} > 0 \text{ are given.}$$

State problem

$$(\mathcal{P}(E)) \qquad u_E \in V: \quad u_E = \arg\inf_{u \in V} \left\{ \frac{1}{2} a_E(u,u) - \int_\Gamma f \cdot u\,ds \right\}, \quad E \in \mathcal{E}$$

Minimum compliance single-load FMO problem

$$(\mathbb{P}) \quad \left\{ \begin{array}{c} \inf\limits_{E \in \mathcal{E}} c(E), \\ \text{subject to: } u_E \text{ satisfies } (\mathcal{P}(E)) \end{array} \right. \qquad c(E) = \int_\Gamma f \cdot u_E\,ds$$

$$\inf_{E\in\mathcal{E}} c(E) = \inf_{E\in\mathcal{E}} \sup_{u\in V} -\Pi(E,u),$$

where

$$\Pi(E,u) = \frac{1}{2} a_E(u,u) - \int_\Gamma f \cdot u \, ds$$

Our goal:

to extend the class of cost functionals and to include also control and state constraints.

Theoretical tools

H-convergence (L. Tartar, F. Murat)

Let $0 < \alpha < \beta$ be given. Define

$$\mathcal{E}^{\alpha,\beta} = \{E \in L^\infty(\Omega, \mathbb{S}^{\bar{N}}) \mid \alpha I_{\bar{N}} \preccurlyeq E \preccurlyeq \beta I_{\bar{N}} \text{ a.e. in } \Omega\}$$

Theorem 1

The set $\mathcal{E}^{\alpha,\beta}$ is H-compact.

Let $\varepsilon > 0$ be given. Define

$$\mathcal{E}^\varepsilon = \{E \in \mathcal{E} \mid E \succcurlyeq \varepsilon I_{\bar{N}} \text{ a.e. in } \Omega\}$$

Proposition 1

The set $\mathcal{E}^\varepsilon$ is H-compact.

Proof.

$\mathcal{E}^\varepsilon \subset \mathcal{E}^{\alpha,\beta}$ for $\alpha = \varepsilon$, $\beta = \bar{\rho}/\bar{N}$. Let $E_n \xrightarrow{\text{H}} E^*$, $E_n \in \mathcal{E}^\varepsilon$.

$$E^* \in \mathcal{E}^\varepsilon \Longleftrightarrow \operatorname{Tr}(E^*) \leq \bar{\rho} \text{ a.e. in } \Omega$$
$$\int_\Omega \operatorname{Tr}(E^*)\, dx \leq \bar{v}$$

$E_n \rightharpoonup \bar{E}$ weakly$*$ and $E^* \preccurlyeq \bar{E}$ a.e. in Ω
$\operatorname{Tr}(\bar{E}) \leq \bar{\rho}$ a.e. in Ω, $\int_\Omega \operatorname{Tr}(\bar{E})\, dx \leq \bar{v}$.

□

Cost functionals

$$J : \mathcal{E}^{\varepsilon} \times V \to \mathbb{R}$$

satisfying

$$\left.\begin{array}{l} E_n \xrightarrow{\text{H}} E,\ E_n, E \in \mathcal{E}^{\varepsilon} \\ v_n \rightharpoonup v \ \text{ in } V \end{array}\right\} \Longrightarrow \liminf_{n\to\infty} J(E_n, v_n) \geq J(E, v) \tag{1}$$

The regularized free material optimization problem

$$\inf_{E \in \mathcal{E}^{\varepsilon}} J(E, u_E), \tag{$\mathbb{P}$}$$

where J satisfies (1) and $u_E \in V$ solves $(\mathcal{P}(E))$.

Theorem 2

Problem $(\mathbb{P})$ has a solution.

Examples of cost functionals satisfying (1)

- the compliance cost functional

$$J(E, u_E) := c(E)$$

- the tracking functional

$$J(E, u_E) := \|u_E - u_0\|^2_{0,\Omega}, \quad u_0 \in V \text{ given}$$

- stress functional

$$J(E, u_E) := \int_\Omega \sigma_E^T \cdot M\sigma_E \, dx,$$

  where M is the von Mises matrix and $\sigma_E = E\varepsilon(u_E)$.

Extension: design dependent functionals

$$\Phi : \mathcal{E}^\varepsilon \to \mathbb{R}, \quad \Phi(E) = \int_\Omega \varphi(E(x))\, dx$$

where $\varphi : \mathbb{S}^{\bar{N}} \to \mathbb{R}$ is monotone:

$$A \preccurlyeq B \Longrightarrow \varphi(A) \leq \varphi(B) \quad A, B \in \mathbb{S}^{\bar{N}} \tag{2}$$

Proposition 2

Let φ be continuous and satisfy (2). If Φ is weakly$*$ lower-semicontinuous, then is also H lower-semicontinuous.

(C. Barbarosie, S. Lopez)

One can add to $\mathcal{E}^{\varepsilon}$ any constraint of the type

$$\Phi(E) \leq C, \quad C \in \mathbb{R} \text{ given},$$

and Theorem 2 still holds.

Extension: state constraints

$$g_I(u_E) \leq C_u, \quad g_{II}(\sigma_E) \leq C_\sigma, \quad C_u, C_\sigma \in \mathbb{R} \text{ given},$$

where g_I, g_{II} are weakly lower-semicontinuous functionals in V and $L^2(\Omega, \mathbb{R}^{\bar{N}})$, respectively.

Define

$$\mathcal{E}^{\varepsilon, g_I, g_{II}} = \{E \in \mathcal{E}^\varepsilon \mid g_I(u_E) \leq C_u,\ g_{II}(\sigma_E) \leq C_\sigma\}$$

Suppose, that $\mathcal{E}^{\varepsilon, g_I, g_{II}} \neq \emptyset$.

Proposition 3

The set $\mathcal{E}^{\varepsilon, g_I, g_{II}}$ is H-compact.

Examples of the state constraints

- linear displacement constraints

$$\int_\Omega d(x) \cdot u_E(x)\, dx, \quad d \in L^2(\Omega, \mathbb{R}^{\bar{N}}) \text{ given}$$

- tracking type displacement constraints

$$\|u_E - u_0\|^2_{0,\Omega} \le C, \quad u_0 \in V \ \text{given}$$

- integral stress constraints

$$\int_\omega \sigma_E^T(x) \cdot M\sigma_E(x)\, dx \le C,$$

where $\omega \subset \Omega$, $M =$ unit or von Mises matrix.

State constrained FMO problem

$$\inf_{E \in \mathcal{E}^{\varepsilon, g_I, g_{II}}} J(E, u_E) \qquad (\mathbb{P})_{g_I, g_{II}}$$

Theorem 3

Problem $(\mathbb{P})_{g_I, g_{II}}$ has a solution.

Discretization of $(\mathbb{P})$ and $(\mathbb{P})_{g_I,g_{II}}$

Two level approach

1^{st} level: discretization of $\mathcal{E}^\varepsilon$ and $\mathcal{E}^{\varepsilon,g_I,g_{II}}$

2^{nd} level: full discretization

Discretization of the design set

$\{S_\kappa\},\ \kappa \to 0_+$ ... system of partitions of $\bar{\Omega}$:

$$\bar{\Omega} = \bigcup_{i=1}^{m} \Omega_i$$

$$\max_i \operatorname{diam} \Omega_i \le \kappa$$

$$\mathcal{E}^\varepsilon_\kappa = \left\{ E \in \mathcal{E}^\varepsilon \,\middle|\, E_i := E|_{\Omega_i} \in (P_0(\Omega_i))^{\bar{N}\times\bar{N}},\ E_i \succcurlyeq \varepsilon I_{\bar{N}},\ \mathrm{Tr}\,(E_i) \le \bar{\rho}\ \forall i,\ \sum_{i=1}^{m} \mathrm{Tr}\,(E_i)|\Omega_i| \le \bar{v} \right\}$$

$$\mathcal{E}_{\kappa}^{\varepsilon,g_I,g_{II}} = \mathcal{E}^{\varepsilon,g_I,g_{II}} \cap \mathcal{E}_{\kappa}^{\varepsilon}$$

1^{st} level approximation of $(\mathbb{P})$ and $(\mathbb{P})_{g_I,g_{II}}$

$$\inf_{E_\kappa \in \mathcal{E}_\kappa^\varepsilon} J(E_\kappa, u) \qquad (\mathbb{P})^\kappa$$

and

$$\inf_{E_\kappa \in \mathcal{E}_\kappa^{\varepsilon,g_I,g_{II}}} J(E_\kappa, u), \qquad (\mathbb{P})^\kappa_{g_I,g_{II}}$$

respectively, where $u \in V$ solves $(\mathcal{P}(E_\kappa))$.

Convergence analysis for $(\mathbb{P})^\kappa$, $\kappa \to 0_+$

Proposition 3

The system $\{\mathcal{E}^\varepsilon_\kappa\}$, $\kappa \to 0_+$ is dense in $\mathcal{E}^\varepsilon$: for any $E \in \mathcal{E}^\varepsilon$ $\exists\{E_\kappa\}$, $E_\kappa \in \mathcal{E}^\varepsilon_\kappa$ such that

$$E_\kappa \to E \quad \text{in } (L^p(\Omega))^{\bar{N}\times\bar{N}} \; \forall p \in [1, \infty) \tag{3}$$

Proof.

$$E_\kappa|_{\Omega_i} = \frac{1}{|\Omega_i|} \int_{\Omega_i} E(x)\,dx$$

□

Corollary

Let $\{E_\kappa\}$ *satisfy* (3). *Then*

$$u_\kappa := u_{E_\kappa} \to u_E \quad \textit{in } V, \; \kappa \to 0_+$$

In addition to (1) suppose that

$$\left.\begin{array}{ll} E_\kappa \to E & \text{in } (L^2(\Omega))^{\bar{N}\times\bar{N}} \\ v_\kappa \to v & \text{in } V, \; \kappa \to 0_+ \end{array}\right\} \Longrightarrow \lim_{\kappa \to 0_+} J(E_\kappa, v_\kappa) = J(E, v) \tag{4}$$

Theorem 4

Let J satisfy (1) and (4). Then from any sequence of optimal pairs $\{(E^*_\kappa, u^*_\kappa)\}$ of $(\mathbb{P})^\kappa$ one can find a subsequence $\{(E^*_{\kappa_j}, u^*_{\kappa_j})\}$ such that

$$\left.\begin{array}{l} E^*_{\kappa_j} \xrightarrow{\text{H}} E^* \\ u^*_{\kappa_j} \rightharpoonup u^* \quad \text{in } V,\ j \to \infty \end{array}\right\} \tag{5}$$

and (E^*, u^*) is an optimal pair of $(\mathbb{P})$. Any accumulation point of $\{(E^*_\kappa, u^*_\kappa)\}$ in the sense of (5) possesses this property.

2^{nd} level: the full discretization

$\kappa > 0$ fixed

$\{V_h\},\ h \to 0_+$ ... a system of finite dimensional subspaces of V with the following density property:

$$\forall v \in V \quad \exists\{v_h\},\ v_h \in V_h : \quad v_h \to v \quad \text{in } V,\ h \to 0_+$$

The Galerkin approximation of $(\mathcal{P}(E_\kappa))$, $E_\kappa \in \mathcal{E}^\varepsilon_\kappa$

$$\left.\begin{array}{l} \text{Find } u_h \in V_h \text{ such that} \\ a_{E_\kappa}(u_h, v_h) = \displaystyle\int_\Gamma f \cdot v_h \, ds \quad \forall v_h \in V_h \end{array}\right\} \qquad (\mathcal{P}(E_\kappa))_h$$

2^{nd} level of the approximation of $(\mathbb{P})^\kappa$

$$\inf_{E_\kappa \in \mathcal{E}^\varepsilon_\kappa} J(E_\kappa, u_h) \qquad (\mathbb{P})^\kappa_h$$

where $u_h \in V_h$ solves $((\mathcal{P}(E_\kappa))_h)$.

Theorem 5

Let J satisfy (4). Then from any sequence $\{(E^*_{\kappa h}, u^*_h)\}$ of optimal pairs of $(\mathbb{P})^\kappa_h$, $h \to 0_+$ one can find a subsequence $\{(E^*_{\kappa h_j}, u^*_{h_j})\}$ such that

$$\left.\begin{array}{l} E^*_{\kappa h_j} \longrightarrow E^*_\kappa \in \mathcal{E}^\varepsilon_\kappa \quad \text{in } (L^\infty(\Omega))^{\bar{N}\times\bar{N}} \\ u^*_{h_j} \longrightarrow u^*_\kappa \quad \text{in } V,\ j \longrightarrow \infty \end{array}\right\} \tag{6}$$

and (E^*_κ, u^*_κ) is an optimal pair of $(\mathbb{P})^\kappa$. Any accumulation point of $\{(E^*_{\kappa h}, u^*_h)\}$ in the sense of (6) possesses this property.

Remark

One can find a filter of indices such that

$$E^*_{\kappa_j h_j} \xrightarrow{\text{H}} E^*, \quad j \longrightarrow \infty$$

where E^* solves $(\mathbb{P})$.

The constrained case ($g_I(u_E) \le 0$, $g_{II}(\sigma_E) \le 0$)

penalty approach $\longrightarrow$ unconstrained case.

A penalty functional $j : \mathbb{R} \to \mathbb{R}$ satisfying:

$$j \in C(\mathbb{R}), \quad j(t) = 0 \ \ \forall t \le 0, \quad t_1 \le t_2 \Rightarrow j(t_1) \le j(t_2)$$

Problem $(\mathbb{P})_{g_I, g_{II}}$ is replaced by

$$\min_{E \in \mathcal{E}^{\varepsilon}} J^{\gamma}(E, u_E), \qquad (\mathbb{P})^{\gamma}$$

where

$$J^{\gamma}(E, u_E) := J(E, u_E) + \frac{1}{\gamma}\big(j(g_I(u_E)) + j(g_{II}(\sigma_E))\big), \quad \gamma \searrow 0_+$$

Proposition 4

Problem $(\mathbb{P})^{\gamma}$ has a solution for any $\gamma > 0$.

Theorem 6

Let $\{(E_j^*, u_j^*)\}$ be a sequence of optimal pairs of $(\mathbb{P})^{\gamma_j}$, $\gamma_j \searrow 0_+$. Then one can find a subsequence $\{(E_{j_k}^*, u_{j_k}^*)\}$ such that

$$\left.\begin{array}{l} E_{j_k}^* \xrightarrow{\mathrm{H}} E^* \in \mathcal{E}^{\varepsilon, g_I, g_{II}} \\ u_{j_k}^* \rightharpoonup u^* \quad \text{in } V, \; k \to \infty \end{array}\right\} \tag{7}$$

Moreover, (E^*, u^*) is an optimal pair of $(\mathbb{P})_{g_I, g_{II}}$. Any accumulation point of $\{(E_j^*, u_j^*)\}$ in the sense of (7) possesses this property.

Example

$$\varepsilon = 10^{-4}, \quad \bar{v} = 0.333|\Omega|, \quad \bar{\rho} = 1$$

cost functional = compliance

Ω = L-shaped structure

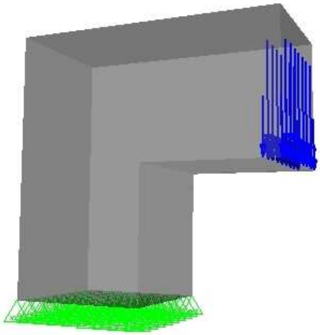
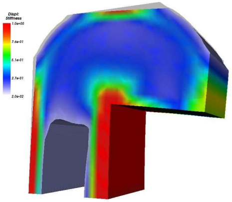


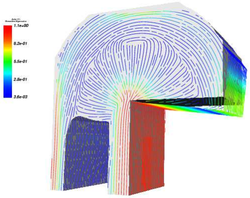
Figure: Geometry and forces.

Example

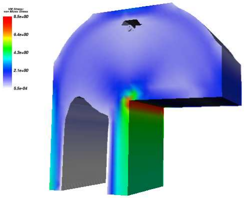
No state constraints



(a) material density



(b) principal material orientation

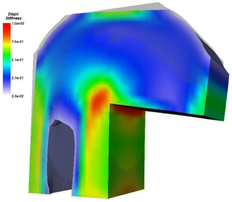


(c) stress distribution

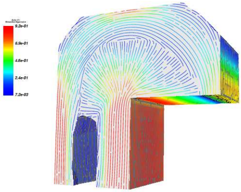
$$J_{opt} = 2.007$$

Example

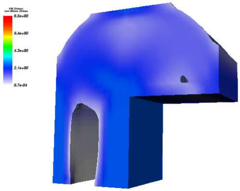
State constraints



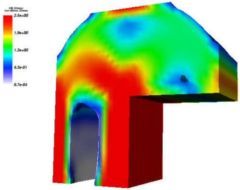
(d) material density



(e) principal material orientation



(f) stress distribution



(g) stress disribution - active set

$$J_{opt} = 2.425$$